Problem Set 7

Note: The problem sets serve as additional exercise problems for your own practice. Problem Set 7 covers materials from §5.1 – §5.3.

In Q1 – Q6, **<u>do not use the Newton-Leibniz formula (or the Fundamental Theorem of Calculus)</u>**.

1. Evaluate each of the following integrals by considering some simple geometric shapes.

   (a) $\displaystyle\int_0^3 f(x)dx$, where $f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 2. \\ 2 & \text{if } 2 \le x \le 3 \end{cases}$

   (b) $\displaystyle\int_0^3 (3 - |x-1| - |x-2|)dx$

   (c) $\displaystyle\int_0^a \sqrt{4-x^2}\,dx$, where $0 \le a \le 2$

   Hint: First sketch the graph of each integrand.

2. Let $I = \int_a^b \sin^2 x\, dx$. Express the following integrals in terms of I.

   (a) $\displaystyle\int_b^a \cos^2 x\, dx$

   (b) $\displaystyle\int_a^b \cos 2x\, dx$

3. Let $a \ge 2$ be a real number.

   (a) By considering an appropriate trapezium, show that

   $$\int_{a-\frac{1}{2}}^{a+\frac{1}{2}} \ln x\, dx \le \ln a.$$

   (b) By considering another appropriate trapezium, show that

   $$\int_{a-1}^{a} \ln x\, dx \ge \frac{\ln(a-1) + \ln a}{2}.$$

4. Let $f(x) = \cos(x^2)$ and $g(x) = \cos(x^3)$. For each of the following pairs of integrals, explain which one has a greater value.

   (a) $\displaystyle\int_0^1 f(x)dx$ and $\displaystyle\int_0^1 g(x)dx$

   (b) $\displaystyle\int_0^1 f(x)dx$ and $\displaystyle\int_{\cos 1}^1 f^{-1}(x)dx$

   (Note that f has an inverse defined on $[\cos 1, 1]$ because $f\colon [0,1] \to [\cos 1, 1]$ is strictly decreasing.)

5. Let $f: [0,1] \to [0, +\infty)$ be a non-negative continuous function such that

$$f(x) \leq x \qquad \text{and} \qquad f(x) \leq 1 - x$$

for every $x \in [0,1]$. Show that

$$\int_0^1 f(x)dx \leq \frac{1}{4}.$$

6. Let $f: [a,b] \to \mathbb{R}$ be a continuous function.
   (a) Show that there exist $c_1, c_2 \in [a,b]$ such that

$$f(c_1)(b-a) \leq \int_a^b f(x)dx \leq f(c_2)(b-a).$$

   (b) Hence deduce that there exists $c \in (a,b)$ such that

$$\int_a^b f(x)dx = f(c)(b-a).$$

7. Evaluate the following antiderivatives.
   (a) $\displaystyle\int (\tan\theta + \cot\theta)^2 d\theta$
   (b) $\displaystyle\int \frac{1}{\sqrt{(2-2x)(2+2x)}}dx$
   (c) $\displaystyle\int 2^{t+2}e^{2t}\,dt$
   (d) $\displaystyle\int \frac{x^4}{1+x^2}dx$
   (e) $\displaystyle\int \frac{3+\sin^2 x}{\cos^2 x}dx$
   (f) $\displaystyle\int \frac{\sin x}{1+\sin x}dx$ (just for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$) *Hint*: Recall Q17(d) of Problem Set 1.

8. (a) Compute

$$\frac{d}{d\theta}\left(\frac{1}{3}\tan^3\theta - \tan\theta\right).$$

   (b) Hence evaluate

$$\int \tan^4\theta\,d\theta.$$

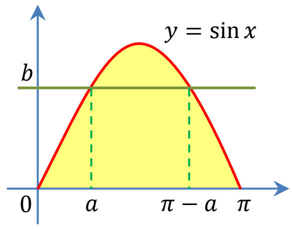
9. Let y and u be variables depending on an independent variable x. It is given that

$$\frac{dy}{dx} = x - k \qquad \text{and} \qquad \frac{du}{dx} = y,$$

where k is a constant. When $x = k$, we have $y = 0$ and $u = 0$.
   (a) Express y in terms of x and k.
   (b) Hence, show that

$$u = \frac{1}{6}(x-k)^3.$$

10. A particle starts moving from rest and its acceleration after t seconds is $(k-2t)\ \mathrm{m/s^2}$, where k is a constant. The velocity of the particle is $12\ \mathrm{m/s}$ after 2 seconds.
    (a) Find the value of k.
    (b) When will the particle change its direction of motion?
    (c) When will the particle return to its original position?
11. A particle starts moving with an initial speed of $16\ \mathrm{m/s}$. It decelerates constantly at a rate of $2\ \mathrm{m/s^2}$. What is its speed after it travels exactly 48 metres?
12. At every point (x,y) on a certain curve in the coordinate plane, we have
$$\frac{d^2y}{dx^2}=6.$$
When $x=2$, y attains a minimum value of 5. Find the equation of the curve.
13. The diagram below shows the graph of
$$f(x)=\sin x$$
on the interval $[0,\pi]$.



    (a) Find the area of the shaded region.
    (b) A horizontal line $y=b$ intersects the graph of f at two points (a,b) and $(\pi-a,b)$, so that the area under the graph of f between the vertical lines $x=a$ and $x=\pi-a$ is exactly half of the whole shaded area. Find the value of b.
14. Evaluate the following integrals.
    (a) $\displaystyle\int_{-1}^{3}|t^3-3t^2+2t|\,dt$
    (b) $\displaystyle\int_{0}^{\frac{\pi}{4}}\frac{1-\sin^3\theta}{\cos^2\theta}\,d\theta$
    (c) $\displaystyle\int_{1}^{\sqrt{3}}\frac{1+x+x^2}{x+x^3}\,dx$

15. (a) Let $p \geq 1$. Show that

$$\frac{1}{(k+1)^p} \leq \int_k^{k+1} \frac{1}{x^p}\, dx \leq \frac{1}{k^p}$$

for every positive integer k. Hence show that

$$\sum_{k=2}^{n+1} \frac{1}{k^p} \leq \int_1^{n+1} \frac{1}{x^p}\, dx \leq \sum_{k=1}^{n} \frac{1}{k^p}$$

for every positive integer n.

(b) Using the result from (a), show that for every positive integer n, we have

(i) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \geq \ln(n+1)$,

(ii) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} < 2$.

16. Evaluate the following antiderivatives, using the substitution rule when necessary.

(a) $\int \csc x\, dx$

(b) $\int \frac{1}{e^x + e^{-x}}\, dx$

(c) $\int \frac{\cos^5 \theta}{\sin^7 \theta}\, d\theta$

(d) $\int \frac{[\ln(u^2)]^2}{u}\, du$

(e) $\int \frac{\sqrt{x}}{1 + x^3}\, dx$